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Classification of extremal double-circulant self-dual codes of length up to 62

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Abstract

All distinct extremal double-circulant self-dual codes of length up to 62 have been found by exhaustive search. These codes are classified in this paper using new and previously known methods. Several of these codes are new extremal self-dual codes.

1. Introduction

A binary $[n,k]$ linear code C is a k-dimensional vector subspace of $\mathrm{GF}(2)^n$, where $\mathrm{GF}(2)$ is the field of two elements. The elements of C are called codewords and the weight of a codeword is the number of non-zero coordinates. An $[n,k,d]$ code is an $[n,k]$ code with minimum (non-zero) Hamming weight d. Two codes are equivalent if one can be obtained from the other by a permutation of coordinates. An automorphism of C is a permutation of the coordinates of C which preserves C. The set of all automorphisms of C forms a group which is called the automorphism group $\mathrm{Aut}(C)$ of C. The dual code $C^{\perp}$ of C is defined as $C^{\perp}=\{x\in \mathrm{GF}(2)^n \mid x\cdot y=0 \text{ for all } y\in C\}$. C is *self-orthogonal* if $C\subseteq C^{\perp}$, and *self-dual* if $C=C^{\perp}$. A code is *doubly even* if all codewords have weight divisible by four, and *singly even* if all weights are even and there is at least one codeword of weight $\equiv 2 \pmod 4$. A self-dual code is *extremal* if it has the largest possible minimum weight for that length.

All self-dual codes of length $n<32$ and all doubly even self-dual codes of length 32 have been classified in [2,3,14,15]. It was shown in [4] that there are exactly three inequivalent extremal singly even [32,16,8] codes. It seems infeasible to classify all self-dual codes for larger lengths. Thus, it is natural to classify self-dual codes under

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some assumption. In this paper, we classify all extremal double-circulant (DC) self-dual codes of length $n \leqslant 62$. In [4] Conway and Sloane presented an improved upper bound for the minimum weight of a self-dual code, and gave a list of possible weight enumerators for extremal self-dual codes of length up to 72. However, the existence of some extremal self-dual codes is still unknown for many of the possible weight enumerators. Recently, several papers have provided constructions for codes whose weight enumerators were not known to exist (cf., e.g. [6–8] and the references given therein). In particular, the first and second authors [6] have constructed extremal singly even $[60, 30, 12]$ codes with a weight enumerator which was not listed in [4]. Several of the codes presented in this paper are new extremal self-dual codes.

All distinct extremal DC self-dual codes of length up to 62 have been found by exhaustive search. Only inequivalent extremal codes are tabulated here due to space limitations. Section 2 presents the methods for constructing DC self-dual codes and classifying such codes. In Sections 3 and 4, the extremal pure and bordered DC singly even codes are classified. Extremal DC doubly even codes are classified in Section 5. Our notation and terminology for coding theory follows that in [13].

2. Preliminaries

2.1. Double-circulant codes

Let D_p and D_b be codes with generator matrices of the form

$$[I \quad R]$$

and

$$\begin{bmatrix} & 0 & 1 & \cdots & 1 \\ & 1 & & & \\ I & \vdots & & R' & \\ & 1 & & & \end{bmatrix},$$

respectively, where I is the identity matrix and R and R' are circulant matrices. An $m \times m$ circulant matrix has the form

$$\begin{bmatrix} r_0 & r_1 & r_2 & \cdots & r_{m-1} \\ r_{m-1} & r_0 & r_1 & \cdots & r_{m-2} \\ \vdots & \vdots & \vdots & & \vdots \\ r_1 & r_2 & r_3 & \cdots & r_0 \end{bmatrix}$$

so that each successive row is a cyclic shift of the previous one. The codes D_p and D_b are called *pure double circulant* and *bordered double circulant*, respectively. These two families of codes are collectively called double-circulant (DC) codes. The bordered

DC construction is used only when $n \equiv 0 \pmod 4$. These constructions have been investigated in several papers (cf., e.g. [5,7,13, Ch. 16]).

MacWilliams [12] presented methods for constructing orthogonal circulant matrices over finite fields. Using these methods, we have constructed all possible extremal DC self-dual codes of length n, $34 \leqslant n \leqslant 62$.

Proposition 2.1 (Gulliver and Harada [7]). *There exists no bordered DC singly even self-dual code of length* $n \equiv 0 \pmod 8$.

Thus, for the case of singly even codes, it is sufficient to find extremal bordered DC codes only for lengths $\equiv 4 \pmod 8$.

2.2. Code equivalence

We now present several methods for checking the equivalence of self-dual codes. Although the following two lemmas are somewhat trivial, they are useful in classifying self-dual codes.

Lemma 2.2. *Let C and C' be self-dual $[2n,n]$ codes with generator matrices of the form $[I_n, A]$ and $[I_n, A^T]$, respectively, where A is an $n \times n$ $(1,0)$-matrix and A^T is the transpose of A. Then C and C' are equivalent.*

Now, let C and C' be binary $[n,k]$ codes. By definition, C and C' are equivalent if and only if there exists a permutation σ of order n such that $C' = C[\sigma]$, where $[\sigma]$ is the $n \times n$ permutation matrix induced by the permutation σ.

Lemma 2.3. *Suppose that the above codes C and C' have generator matrices of the form $G = [g_1, \ldots, g_n]$ and $G' = [I_k, B]$, respectively, where g_i is the ith column of G and B is a k by $(n-k)$ $(1,0)$-matrix. Then C and C' are equivalent if and only if there exists an injection*

$$\rho : \{1,2,\ldots,k\} \to \{1,2,\ldots,n\}$$

with the following property: Let $T = [g_{\rho(1)}, \ldots, g_{\rho(n)}]$. Then there exists an injection

$$\tau : \{k+1,\ldots,n\} \to \{1,2,\ldots,n\} \backslash \operatorname{Im} \rho$$

such that $TB = [g_{\tau(k+1)}, \ldots, g_{\tau(n)}]$ where $\operatorname{Im} \rho$ denotes the image of ρ.

Proof. Note that $C' = C[\sigma]$ if and only if there exists a $k \times k$ matrix T such that $TG' = G[\sigma]$. If $C' = C[\sigma]$, then $TG' = G[\sigma]$ for some square matrix T. Since $TG' = [T, TB]$ and $G[\sigma] = [g_{\sigma(1)}, \ldots, g_{\sigma(n)}]$, $\rho = \sigma|_{\{1,\ldots,k\}}$ and $\tau = \sigma|_{\{k+1,\ldots,n\}}$ are the desired injections.

Consequently, if there exist injections ρ and τ as above, the following permutation σ of order n can be defined

$$\sigma(i)=\begin{cases}\rho(i), & 1\leqslant i\leqslant k,\\ \tau(i), & k+1\leqslant i\leqslant n.\end{cases}$$

It is clear that if $TG'=G[\sigma]$ then $C'=C[\sigma]$. □

We now present a method to check the inequivalence of self-dual codes. Let C be a self-dual $[2n,n,d]$ code. Let $M_w=(m_{ij})$ be an A_w by $2n$ matrix whose rows are the codewords of weight w in C. For an integer k $(1\leqslant k\leqslant 2n)$, let $n_w(j_1,\ldots,j_k)$ be the number r $(1\leqslant r\leqslant A_w)$ such that $m_{rj_1}\cdots m_{rj_k}\neq 0$ for $1\leqslant j_1<\cdots<j_k\leqslant 2n$. We consider the set

$$S_w=\{n_w(j_1,\ldots,j_k)\mid k \text{ distinct columns } j_1,\ldots,j_k\}.$$

Let $M_w(k)$ and $m_w(k)$ be the maximal and minimal numbers in S_w, respectively. For convenience, denote $M_d(k)$ and $m_d(k)$ by $M(k)$ and $m(k)$, respectively. These numbers are invariant under the equivalence of binary codes (as well as ternary and other codes). In [9], this method was employed to classify ternary extremal self-dual codes.

Since two codes are inequivalent if their automorphism groups are non-isomorphic, this property will also be used in this paper to classify extremal self-dual codes.

2.3. *Extremal weight enumerators*

For the lengths of interest in this paper, the weight enumerators of extremal doubly-even codes are unique, but there are many possible weight enumerators for extremal singly even codes (cf. [4]). Therefore, we list below the weight enumerators for which extremal DC self-dual codes may exist. The weight enumerators for some extremal DC codes have been determined in [7]. Note that since no extremal DC code of length 62 exists, we omit W_{62}.

Length 34: $W_{34}=1+(34-4\beta)y^6+(255+4\beta)y^8+\cdots$,
Length 36: $W_{36}=1+289y^8+1632y^{10}+\cdots$,
Length 38: $W_{38}=1+171y^8+1862y^{10}+\cdots$,
Length 40: $W_{40}=1+(125+16\beta)y^8+(1664-64\beta)y^{10}+\cdots$,
Length 44: $W_{44}=1+(44+4\beta)y^8+(976-8\beta)y^{10}+\cdots$,
$W'_{44}=1+(44+4\beta)y^8+(1232-8\beta)y^{10}+\cdots$,
Length 46: $W_{46}=1+1012y^{10}+9660y^{12}+\cdots$,
Length 48: $W_{48}=1+768y^{10}+8592y^{12}+\cdots$,
Length 50: $W_{50}=1+(580-32\beta)y^{10}+(7400+160\beta)y^{12}+\cdots$,
Length 52: $W_{52}=1+(442-16\beta)y^{10}+(6188+64\beta)y^{12}+\cdots$,
$W'_{52}=1+250y^{10}+7980y^{12}+\cdots$,
Length 54: $W_{54}=1+(351-8\beta)y^{10}+(5031+24\beta)y^{12}+\cdots$,

Table 1
The weight enumerators of pure double-circulant codes

Length	Number	Weight enumerator	Length	Number	Weight enumerator
34	16	W_{34} $(\beta=0)$	46	22	W_{46}
38	2	W_{38}	48	8	W_{48}
40	8	W_{40} $(\beta=0)$	50	40	W_{50} $(\beta=0)$
	14	W_{40} $(\beta=10)$	52	72	W_{52} $(\beta=0)$
44	20	W'_{44} $(\beta=0)$	54	162	W_{54} $(\beta=0)$
	10	W'_{44} $(\beta=22)$	58	84	W_{58} $(\beta=\gamma=0)$
	10	W'_{44} $(\beta=44)$		160	W_{58} $(\beta=0,\gamma=54)$
	2	W'_{44} $(\beta=154)$	60	8	W_{60} $(\beta=10)$

Length 58: $W_{58}=1+(319-24\beta-2\gamma)y^{10}+(3132+152\beta+2\gamma)y^{12}+\cdots$,
Length 60: $W_{60}=1+(2555+64\beta)y^{12}+(33\,600-384\beta)y^{14}+\cdots$,
$W'_{60}=1+3451y^{12}+24\,128y^{14}+\cdots$.

3. Pure double-circulant singly even codes

In this section, the extremal pure DC singly even codes are classified. First, an algebraic consideration yields the following proposition.

Proposition 3.1 (Gulliver and Harada [7]). *There exists a unique extremal pure double-circulant self-dual code of length n for* $n=38$ *and* 46.

Table 1 presents the weight enumerators and the corresponding numbers of codes.

There are two extremal pure DC $[44,22,8]$ codes in this table with $\beta=154$ in W'_{44}. From Lemma 2.2, these codes are equivalent. For other lengths, the methods in Section 2 have been used to determine the equivalent codes. Table 2 gives the first row of R, the weight enumerator and the order $|\mathrm{Aut}(P_{n,i})|$ of the automorphism group, and the values of $M(2)$ and $m(2)$ for a representative code from each equivalence class.

The three pairs of codes $P_{44,1}$ and $P_{44,2}$, $P_{54,3}$ and $P_{54,6}$, and $P_{54,8}$ and $P_{54,9}$ in Table 2 have identical values for $M(2)$ and $m(2)$. For these, we obtained the following values for $(M_{10}(4),m_{10}(4)),(9,0)$ and (7,0), (19,0) and (21,0), and (21,0) and (22,0), respectively. This establishes that these codes are inequivalent.

4. Bordered double-circulant singly even codes

In this section, we give all inequivalent extremal bordered DC singly even codes of length n, $36\leqslant n\leqslant 60$. From Proposition 2.1, we need only consider codes of length $\equiv 4 \pmod 8$. Table 3 presents the weight enumerators and the corresponding numbers of codes.

Table 2
The pure double-circulant codes

Length	Code	First row of R	Weight enumerator	$M(2)$	$m(2)$	$\lvert\mathrm{Aut}(P_{n,i})\rvert$
34	P_{34}	000001110011011111	W_{34}			34
38	P_{38}	00001010111100100 11	W_{38}			342
40	$P_{40,1}$	0000010001101101 0111	W_{40} $(\beta=0)$			40
	$P_{40,2}$	0000001010011111 0101	W_{40} $(\beta=10)$	41	5	327 680
	$P_{40,3}$	0000100010011011101		29	5	20 480
	$P_{40,4}$	0000100010111001101		33	9	1 474 560
	$P_{40,5}$	0001000100111001 1011		25	1	44 236 800
	$P_{40,6}$	0001001000110011 0111		25	7	11 796 480
44	$P_{44,1}$	00000001100101001011 11	W'_{44} $(\beta=0)$	5	0	44
	$P_{44,2}$	00001011010110011111 11		5	0	44
	$P_{44,3}$	00001010011011111 10111	W'_{44} $(\beta=22)$			44
	$P_{44,4}$	00000000011100011 10111	W'_{44} $(\beta=44)$			44
	$P_{44,5}$	01010111011111011 11111	W'_{44} $(\beta=154)$			$2^{16}3^4 5^2 7^2 11^2$
46	P_{46}	00000000100111100 110101	W_{46}			46
48	P_{48}	00000001010010010 0011111	W_{48}			48
50	$P_{50,1}$	00000001000111000 11001101	W_{50} $(\beta=0)$	28	12	50
	$P_{50,2}$	00010011111101111 11101011		28	14	50
52	$P_{52,1}$	00000000101011011 11111 10011	W_{52} $(\beta=0)$	29	7	52
	$P_{52,2}$	00000001000111110 11101 0111		24	8	52
	$P_{52,3}$	00000001001010111 11111 10011		22	6	52
	$P_{52,4}$	00000011101011100 11000 1111		22	9	52
	$P_{52,5}$	00000100010100011 11111 10111		25	6	52
	$P_{52,6}$	00001000010101101 10111 1101		24	7	52
54	$P_{54,1}$	00100001010110011 11111 11111	W_{54} $(\beta=0)$	17	4	54
	$P_{54,2}$	00010110000000100 10111 11111		26	4	54
	$P_{54,3}$	00010001000011000 11011 11111		19	5	54
	$P_{54,4}$	00001000001101100 10011 11111		20	6	54
	$P_{54,5}$	00101000101000001 10101 11111		20	4	54
	$P_{54,6}$	00001000100100011 11001 11111		19	5	54
	$P_{54,7}$	00000110110100000 11100 11111		21	5	54
	$P_{54,8}$	01000000001000000 10001 01111		16	6	54
	$P_{54,9}$	00000000001010010 00011 01111		16	6	54
58	$P_{58,1}$	00001110100101010 01011 11111 11	W_{54} $(\beta=\gamma=0)$	16	4	58
	$P_{58,2}$	00010101001010000 01001 11111 11		21	2	58
	$P_{58,3}$	00000000100100000 00101 00111 11		18	0	58
	$P_{58,4}$	00000111011010101 11111 10011 011	W_{54} $(\beta=0,\ \gamma=54)$	10	2	58
	$P_{58,5}$	00000111111011011 10000 11101 1		12	0	58
	$P_{58,6}$	00000111101110100 01101 00010 01		12	1	58
	$P_{58,7}$	00001001001111101 10111 10111 01		12	2	58
	$P_{58,8}$	00001000110100101 11001 01101 01		14	0	58
	$P_{58,9}$	00000111110011110 01110 11001 11		14	1	58
	$P_{58,10}$	00000111100100111 10011 11111 01		15	0	58
	$P_{58,11}$	00011011101111111 10111 11001 11		21	3	406
60	P_{60}	00000001000000100 11111 11110 11101	W_{60} $(\beta=10)$			60

Table 3
The weight enumerators of bordered double-circulant codes

Length	Number	Weight enumerator
36	16	W_{36}
44	24	$W_{44}(\beta = 17)$
	30	$W_{44}(\beta = 38)$
52	40	$W_{52}(\beta = 0)$
60	84	W'_{60}

Table 4
The bordered double-circulant codes

Length	Code	First row of R'	W	$\lvert\mathrm{Aut}(B_{n,i})\rvert$	$M(2)$	$m(2)$
36	B_{36}	00000001110101111	W_{36}	34		
44	$B_{44,1}$	00000110001111111011	W_{44} $(\beta=17)$	42	7	0
	$B_{44,2}$	00000111100011011111		42	7	1
	$B_{44,3}$	00000000100110101011	W_{44} $(\beta=38)$	42	28	1
	$B_{44,4}$	00000100100011101011		42	28	1
	$B_{44,5}$	00001011011110100111		84	28	1
52	$B_{52,1}$	00000000010011001011000011	W'_{52}	50	24	0
	$B_{52,2}$	00010011110111011111100111		50	24	0
60	$B_{60,1}$	00000000010110101011010001111	W'_{60}	58	319	55
	$B_{60,2}$	00000001101111110101101010111		58	319	55
	$B_{60,3}$	00000110101111111011011111111		58	319	55

For each length, we complete the classification by listing in Table 4 the first row of R', the weight enumerator W, the order $|\mathrm{Aut}(B_{n,i})|$ of the automorphism group and the values of $M(2)$ and $m(2)$ for a representative code from each equivalence class. For codes $B_{44,3}$ and $B_{44,4}$ listed in the table, $(|\mathrm{Aut}(B_i)|, M(2), m(2))$ are identical. However the values of $(M(3), m(3))$ are (6,0) and (4,0), respectively, so these two codes are inequivalent. Similarly, the values of $(M(3), m(3))$ are (55,4), (55,2) and (55,0) for codes $B_{60,1}$, $B_{60,2}$ and $B_{60,3}$, respectively. Thus these three codes are inequivalent. For codes $B_{52,1}$ and $B_{52,2}$, $(M(2), m(2), M(3), m(3), M(4), m(4))$ are identical. However, the values of $(M_{10}(3), m_{10}(3))$ are (14,0) and (12,0), respectively, establishing that these two codes are also inequivalent.

5. The double-circulant doubly even codes

In this section, the extremal DC doubly even codes of lengths 40, 48 and 56 are classified. Table 5 gives the number N_p (resp. N_b) of distinct extremal pure (resp. bordered) DC doubly-even codes.

Table 5
The number of double-circulant doubly even codes

Length	N_p	N_b
40	46	26
48	8	8
56	0	162

Table 6
The double-circulant codes of length 40

	Type	First row of R or R'	$M(2)$	$m(2)$
$C_{40,1}$	Bordered type	0000000010010110101	19	1
$C_{40,2}$	Bordered type	0000010011111100111	38	2
$C_{40,3}$	Bordered type	0000101011110011011	12	0
$C_{40,4}$	Pure type	00000000011011000111	22	1
$C_{40,5}$	Pure type	00000000100110001111	12	0
$C_{40,6}$	Pure type	0000010101111101011	41	5
$C_{40,7}$	Pure type	0000100111101011011	21	3
$C_{40,8}$	Pure type	0000101110111101011	33	9
$C_{40,9}$	Pure type	0000101111101011011	20	2
$C_{40,10}$	Pure type	0001000101111011011	29	5
$C_{40,11}$	Pure type	0001001100111011011	25	7
$C_{40,12}$	Pure type	0001100100111011011	25	1

5.1. *Codes of length* 40

It follows from the form of the generator matrix that a bordered DC [40,20] code has an automorphism of order 19. We have found at least three inequivalent extremal bordered DC codes using the method described in Section 2. The first row of R', as well the values of $M(2)$ and $m(2)$, are listed in Table 6. There exist exactly three inequivalent extremal doubly even self-dual [40,20,8] codes with automorphisms of order 19 (cf. [16]). Thus there exist exactly three inequivalent extremal doubly even bordered DC [40,20,8] codes.

By computer, we have verified that there exist exactly nine inequivalent extremal pure DC doubly even codes of length 40. The first row of R for these codes, as well as the values of $M(2)$ and $m(2)$, are listed in Table 6. Among these nine codes, only $C_{40,5}$ has an automorphism of order 19, and this code is equivalent to $C_{40,3}$. Thus, there exists 11 inequivalent extremal doubly even DC codes of length 40.

5.2. *Codes of length* 48

Let P_{48} (resp. B_{48}) be an extremal doubly even pure (resp. bordered) DC code of length 48. It can easily be shown that P_{48} (resp. B_{48}) has an automorphism of order

Table 7
The double-circulant codes of length 56

	First row of R'	$M(4)$	$m(4)$	$\lvert\mathrm{Aut}(C_{56,i})\rvert$
$C_{56,1}$	00000000000011001010111011	23	1	54
$C_{56,2}$	00000000101101111111010111	26	2	54
$C_{56,3}$	00000001001110011101110111	25	1	54
$C_{56,4}$	00000001101100100111101111	24	2	54
$C_{56,5}$	00000010100111110101101011	26	0	27
$C_{56,6}$	00000011000011101101101111	25	1	27
$C_{56,7}$	00000110001111101001011111	24	1	27
$C_{56,8}$	00001001110111110111111101	25	2	27
$C_{56,9}$	00001001111011010111111111	28	2	27

3 (resp. 23). It follows from the main theorem in [10] that P_{48} and B_{48} are equivalent to the extended quadratic residue code. Thus all extremal DC doubly even codes of length 48 are equivalent.

5.3. *Length* 56

Using the method given in Section 2, we have found that there are at most nine equivalence classes of extremal bordered DC codes of length 56. Since the codewords of minimum weight in an extremal doubly even code of length 56 yield a 3-design, $M(1), m(1), M(2), m(2), M(3)$ and $m(3)$ are identical for these codes. Therefore, we compare $M(4)$ and $m(4)$ to determine the inequivalent codes. The first row of $R', M(4), m(4)$ and the order of the automorphism group for the nine representative codes are listed in Table 7. Only $C_{56,3}$ and $C_{56,6}$ in the table have the same values of $M(4)$ and $m(4)$. Since these two codes have different automorphism groups, it follows that there exists nine inequivalent extremal doubly even bordered DC [56, 28, 12] codes.

These nine codes are now compared with the known extremal codes. It was proved in [17] that there are exactly 16 extremal doubly even [56, 28, 12] codes with automorphisms of order 13. Four inequivalent extremal codes have also been constructed from Hadamard matrices of order 28 (cf. [1, 11]). Moreover, the first author [8] constructed 138 extremal codes and defined K-matrices in order to distinguish these codes. In [8] the question of the equivalence of the No. 2 code contained therein with the other 157 codes was left open. Thus, it is known that there are at least 157 inequivalent extremal doubly even codes of length 56. Since our computer results show that the No. 2 code in [8] and $C_{56,8}$ in Table 7 are equivalent, this code has no automorphism of order 13. Therefore, it is not equivalent to the other 157 codes in [8]. Moreover, we have determined that the codes given in this paper, with the exception of $C_{56,8}$, are not equivalent to the 158 known codes by comparing their K-matrices of degree 22. Thus, there exists at least 166 not equivalent extremal doubly even codes of length 56.

Note that there exist very many extremal doubly even [40, 20,8] codes (cf., e.g. [8]), so that in this case we have not checked the equivalence of our codes with known codes.

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